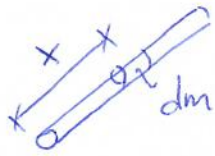
DYNAMICS OF SYSTEMS OF PARTICLES

Center of mass

- of discrete body: $M\bar{r}_{CM} = \sum m_i \bar{r}_i$
- of continuous body: $M\bar{r}_{CM} = \iiint \bar{r}\, dm$

Concept: $x_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{M_T}$

Example: $\lambda = \dfrac{M}{L} = \dfrac{dm}{dl} \Rightarrow dm = \lambda\, dx$



$$M x_{CM} = \int x\, dm = \int_0^L x \lambda\, dx = \lambda \left[\frac{x^2}{2}\right]_0^L$$

$$M x_{CM} = \lambda \frac{L^2}{2} = \frac{M}{L}\frac{L^2}{2} \Rightarrow \boxed{x_{CM} = \frac{L}{2}}$$

Potential Energy

The same energy as if all mass may be concentrated in the CM

$$U = M_T g h_{CM}$$

Motion of CM

$$M \frac{d\bar{r}_{CM}}{dt} = \sum m_i \frac{d\bar{r}_i}{dt} = \sum m_i \bar{v}_i$$

$$M \bar{a}_{CM} = \sum F_i = \sum m_i a_i = F_T$$

"The CM moves like a particle of mass M_T subject to F_T"

Conservation of linear momentum

"one of the most important laws!"

For a particle: $\bar{p} = m\bar{v}$

$$\frac{d\bar{p}}{dt} = \frac{d(m\bar{v})}{dt} = m\bar{a}$$

$$\bar{F} = \frac{d\bar{p}}{dt}$$

For a system:

$$\bar{P} = \Sigma p_i = \Sigma m_i v_i$$

$$\bar{P} = M\bar{v}_{CM}$$

$$\boxed{\frac{d\bar{P}}{dt} = M\bar{a}_{CM}}$$

$$\boxed{\bar{P} = \text{constant} \Leftrightarrow \Sigma F_i = F_{net} = \bar{0}}$$

Remark: $\bar{P}$ can be conserved, this does not imply energy is conserved.

$$m_0 \cdot v_0 + 0 = (m_0 + m_{water})\, v_f$$

$\bar{P}$ is constant.

$\begin{cases} E_p = E = mgh \text{ is constant? But then } m_f = m_0 + m_{water} \\ E_c = T = \frac{1}{2}mv^2 \text{ changes!} \end{cases}$

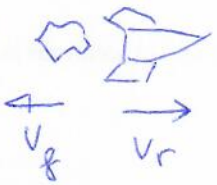
"Interactions ≡ collisions"

- Elastic collision: $\bar{P}$ constant, E constant
- Inelastic collision:
  - complete → the particles remain stuck together
  - partial → general case.

Systems of variable mass: i.e. rockets

— Simple one-stage rocket.

$$p_0 = (m_{fuel} + m_{rocket}) \cdot 0 = 0$$



$$p_f = 0 = m_{rocket} \cdot v_r + m_{fuel} \cdot v_{fuel}$$

In a real case, not all the fuel is exhausted in one-shot → mass changes continously:

$$p(t) = m(t)v(t) = mv$$

$R \equiv$ fuel exhaust rate $= \frac{dm}{dt}$

$$p(t+\Delta t) = \underbrace{(m - R\Delta t)(v+\Delta v)}_{\text{rocket}} + \underbrace{R\Delta t(v - v_{exhaust})}_{\text{gas}}$$

$$p(t+\Delta t) = mv + m\Delta v - v_{exh} R \Delta t + \underbrace{\Delta t \cdot \Delta v}_{\simeq 0} \cdots$$

$$\frac{\Delta p}{\Delta t} = \frac{p(t+\Delta t) - p(t)}{\Delta t}$$

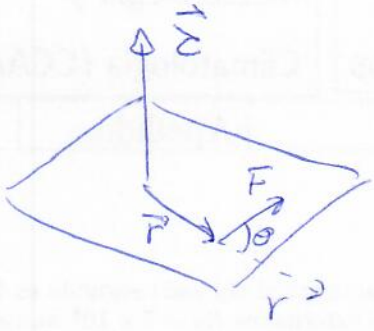
$$\begin{cases} \frac{\Delta p}{\Delta t} = m \frac{\Delta v}{\Delta t} - v_{exh} \cdot R \\ \frac{\Delta p}{\Delta t} = F_{ext\,net} = \sum F = -mg \end{cases}$$

$$\Rightarrow \boxed{m \frac{dv}{dt} = R v_e - mg}$$

Eq for a simple rocket.-

Rotations and Torques

For one particle:



$$\vec{\tau} = \bar{r} \times \bar{F}$$

Torques imply rotations

↳

$$ds = r\, d\theta$$

$$\omega = \frac{d\theta}{dt} \qquad \alpha = \frac{d^2\theta}{dt^2}$$

$$\bar{\tau} = \bar{r} \times \bar{F}$$

$$|\bar{\tau}| = |r|\,|F|\sin\theta$$

$$|\bar{\tau}| = |r|\,|F_{tangential}|$$

$$|\tau| = m a_t \cdot r$$

$$a_t = \alpha \cdot R$$

$$\boxed{|\tau| = m\,\alpha\, r^2}$$

For a system of particles

$$\sum \tau_i = \sum m_i r_i^2 \alpha$$

Let's define

MOMENT OF INERTIA → $I \equiv \sum m_i r_i^2 \Rightarrow I \equiv \int r^2 dm$

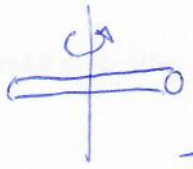
discrete — continuous.

$$\boxed{\sum \tau_i = I\alpha}$$ ← similar to $F = ma^2$ but for rotations

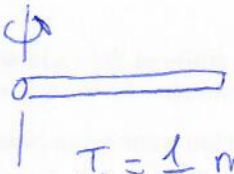
Torques ⇒ rotations !

$$I = \int r^2 dm$$

it depends on distance to rotation axis, hence on the axis



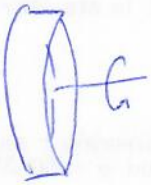
$I = \frac{1}{12} m\ell^2$



$I = \frac{1}{3} m\ell^2$

$I = \frac{1}{2} mR^2$

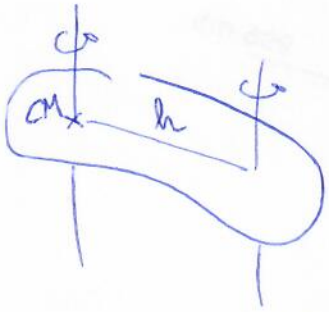
$I = \frac{2}{3} MR^2$



$I = mR^2$

$I = \frac{2}{5} MR^2$

Parallel axis theorem



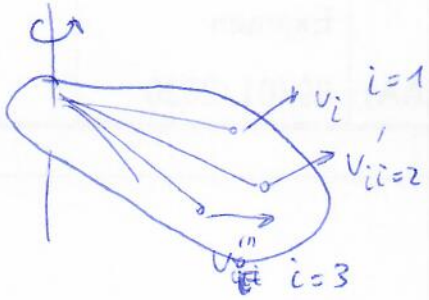
$$I = I_{CM} + Mh^2$$

Example:

$$I_B = I_{CM} + M\left(\frac{\ell}{2}\right)^2$$

$$= \frac{1}{12} m\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{12} m\ell^2 + \frac{1}{4} m\ell^2 = \frac{1}{3} m\ell^2$$

Angular momentum and rotations



$$\vec{L} \equiv \vec{r} \times \vec{p}$$

$$L_i = r_i \times m_i v_i$$

$$L_i = m v_i r_i \vec{k}$$

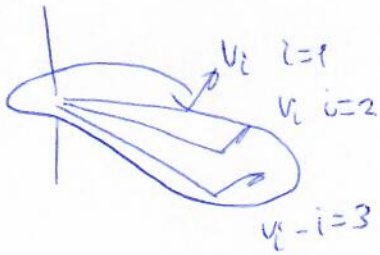
$$L_i = m r_i^2 \omega \vec{k}$$

$$L_T = \Sigma L_i = \Sigma m_i r_i^2 \omega \vec{k} = I \vec{\omega}$$

$$\boxed{\vec{L}_T = I \vec{\omega}}$$

If $\vec{\tau} = 0 \Rightarrow \vec{L}$ is constant $\Rightarrow I\omega$ is constant

Kinetic energy and rotations



$$E_c = T_i = \frac{1}{2} m_i v_i^2$$

$$\Sigma E_{c_i} = \Sigma T_i = T_{\text{total rotation}}$$

$$T_{rot} = \frac{1}{2} (\Sigma m_i r_i^2) \omega^2$$

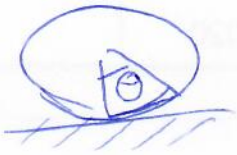
$$T_{rot} = \frac{1}{2} I \omega^2$$

Work and power

$$dW_i = F_i \, ds_i = F_i \, r_i \, d\theta = \tau_i \, d\theta$$

$$P = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \boxed{\tau \omega}$$

Rolling without sliding

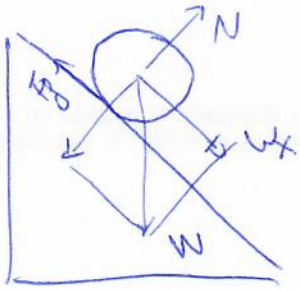


$$s = R\theta$$

$$V_{cm} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$\left.\begin{aligned} V_{CM} &= R\omega \\ a_{CM} &= R\alpha \end{aligned}\right| \rightarrow \text{non-slip condition}$$

$$T_T = \underbrace{\frac{1}{2} I_{CM}\omega^2}_{\text{rotation}} + \underbrace{\frac{1}{2} M_T V_{CM}^2}_{\text{traslation}}$$



$$w_x - F_f = m a_{cm}$$

$$mg\sin\theta - \frac{I}{R} = m a_{CM}$$

$$mg\sin\theta - \left(\underbrace{I_{CM}\cdot\frac{a_{CM}}{R^2}}_{\tau = I\cdot\alpha = I\frac{a}{R}}\right) = m a_{CM}$$

$$a_{CM} = \frac{1}{1 + \frac{I_{CM}}{mR^2}} g\sin\theta$$